Games in the View of Mathematics

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Sheets can be downloaded from

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Curriculum vitae

1985	**Ph.D.** University of Bonn / Max-Planck-Institut für Mathematik Number theory, algebraic topology
1989-	**MEGA-Spielgeräte**, Limburg, www.mega-spiel.de (managing director) Development and distribution of AWPs (“Amusement with prices”) based on German regulations
1992-	**GeWeTe**, Mechernich, www.gewete.de (director of R+D) Money changer
1999-	**Mega Web**, Limburg, www.megaweb-online.de (managing director) Public internet terminals
1998	**Book:** „Glück, Logik und Bluff, Mathematik im Spiel: Methoden, Ergebnisse und Grenzen“ (“Chance, logic and bluff: Mathematics in games: methods, results and limits”), Braunschweig (Vieweg) 1998

Games in the view of mathematics

- Have I a **chance to win** and **how big** is it?
- Which is the **best move** for me?
- **Are** the possible **moves comparable** in an objective manner?
- How to **program a computer playing** chess, back-gammon, poker etc.?
- Is there a relation between the mathematical character of a game and the character which a game has in the view of it's players?

Why do we play?

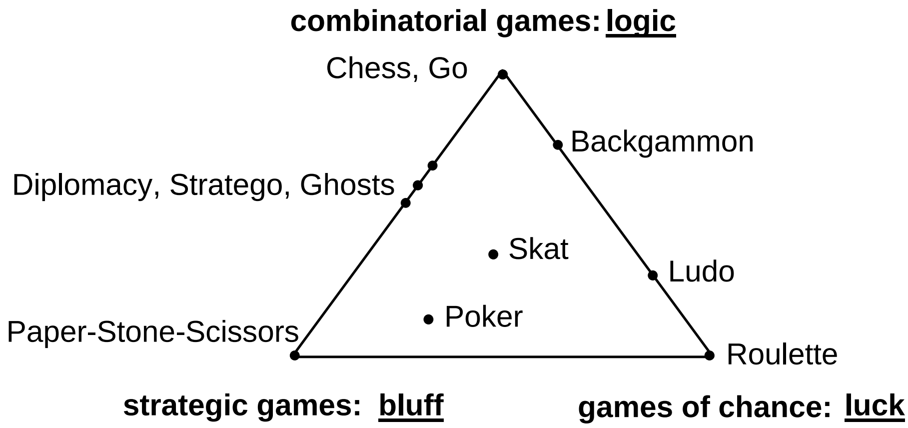
- amusement, thrill and the hope to win

Where are they coming from?

- uncertainty – course and result of a game

Reasons for uncertainty

- randomness
- combinatorial multiplicity
- imperfect information



Mathematical disciplines:

Games of chance:

- probability theory
- statistics

Combinatorial games:

- combinatorial game theory
- complexity theory, algorithmic
- game theory

Strategic games:

- game theory

Games of chance – the mathematics

Probability: a measure for events of random experiments (e.g. the chance to win a game):

- repeat an experiment very often:

$$\frac{\textbf{number of experiments with observed success}}{\textbf{total number of experiments}} \xrightarrow{\textbf{trend}} \textbf{probability}$$

  Law of large numbers: there is a secure trend for relative frequencies, but there isn't any tendency to balance absolute frequencies

- secure event has probability 1,
- impossible event has probability 0,
- events which are symmetric to each other have the same probability, e.g. $^1/_6$ for the six events of a die

Random variable: e.g. the amount of a win

Expectation: e.g. the “fair” stake of a game

E(X + Y) = E(X) + E(Y)
“fair“ stake for two games of chance X, Y;

E(X·Y) = E(X)·E(Y) for **independent** X,Y;
”fair“ stake if the win X of the first game is used as stake of a the second (independent) game Y

attention: Petersburg paradox, 1713,

Early history of probability theory: Games of chance

1222-1268	Anonymous: combinatorics of 3 dice and inter-pretation of the results as „chance“
ca. 1400	Anonymous: problem of division of the stake: How the stake is to be divided if a series of a game of chance is broken before it’s end
later	several wrong solutions of the first two problems
1654	**Fermat**, **Pascal** (and **de Méré**) solved: ♦ problem of division of stake ♦ Four throws with one die are enough to make a good bet on throwing at minimum one “6”. But: Are 24 throws enough to make a good bet on throwing 6-6 at minimum once? Answer: “No”!
...	**Huygens**, **Laplace**, **Bernoulli** et.al.: In their probability research games of chance are used as standard examples beside others (because they are well determined model situations)

Monopoly

Compare the return of investments between the different groups of streets! Program a simulation or compute:

- For each square ("street"):
  expected income = rent per visit × probability of visit
- How to compute such a probability of "visit"?

	Street (German edition)	Prob.	max. rent (hotel etc.) rent	expec.	group
0	Los	0,02889			
1	Badstr.	0,02436	5000	122	
2	Gemeinschaftsfeld	0,01763			
3	Turmstr.	0,02040	9000	184	305
4	Einkommenssteuer	0,02210			
5	Südbahnhof	0,02686	4000	107	
6	Chausseestr.	0,02169	11000	239	
7	Ereignisfeld	0,00972			
8	Elisenstr.	0,02246	11000	247	
9	Poststr.	0,02217	12000	266	752
10	Nur zu Besuch	0,02184			
11	Seestr.	0,02596	15000	389	
12	Elektrizitätswerk	0,02378	1400	33	
13	Hafenstr.	0,02213	15000	332	
14	Neue Str.	0,02457	18000	442	1164
15	Westbahnhof	0,02531	4000	101	
16	Münchener Str.	0,02703	19000	514	
17	Gemeinschaftsfeld	0,02306			
18	Wiener Str.	0,02821	19000	536	
19	Berliner Str.	0,02794	20000	559	1608
20	Frei parken	0,02806			
21	Theaterstr.	0,02594	21000	545	
22	Ereignisfeld	0,01209			
23	Museumsstr.	0,02549	21000	535	
24	Opernplatz	0,02983	22000	656	1736
25	Nordbahnhof	0,02718	4000	109	
26	Lessingstr.	0,02540	23000	584	
27	Schillerstr.	0,02521	23000	580	
28	Wasserwerk	0,02480	1400	35	68
29	Goethestr.	0,02441	24000	586	1750
30	Gefängnis	0,09422			
31	Rathausplatz	0,02501	25500	638	
32	Hauptstr.	0,02438	25500	622	
33	Gemeinschaftsfeld	0,02193			
34	Bahnhofstr.	0,02312	28000	647	1907
35	Hauptbahnhof	0,02243	4000	90	407
36	Ereignisfeld	0,00934			
37	Parkstr.	0,02023	30000	607	
38	Zusatzsteuer	0,02023			
39	Schloßallee	0,02457	40000	983	1590

In some minor details of the game (concerning transfers from "Chance" and "Community Chest") the computation is based on the German edition

- Monopoly is equivalent to regular Markov chain of 120 states. So there exists exact one equilibrium.
- Long term probabilities can be computed using a system of 121 linear equations and 120 variables
- P("Opernplatz") = 1.48 × P("Parkstraße")

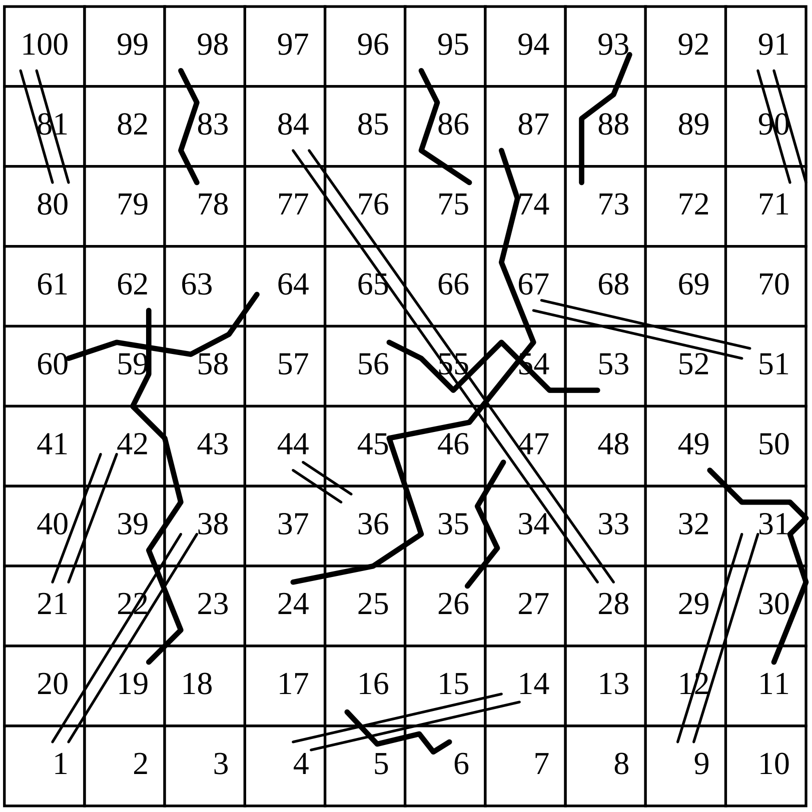
Snakes and ladders

- “Snakes and ladders” is a pure game of chance (a player can’t decide anything): Each player moves a man. The number of fields the man is moved forward is determined by throwing a single die.
- “Snakes and ladders” is equivalent to an absorbing Markov chain with one absorbing state (the “goal”).
- **How big is the expected number of rolls to reach the goal?**

  Iterate the transfer formula for the probability distribution or use a direct formula based on

  $$(I - Q)^{-1},$$

  Q being a block matrix consisting on the transfer probabilities of the non absorbing states.



ladders: go upwards **snakes: go downwards**

Video-Poker

Rules of the one-person game (as used in several thousands of video slots in Las Vegas):

- ♦ A deck of 52 cards is used.
- ♦ The player first draws 5 cards.
- ♦ Now he can select 0 to 5 cards to hold, the others are removed and replaced by randomly drawn cards from the rest of 47 cards.
- ♦ After the second draw of cards the game ends. The payoff is depending on the combination of the 5 cards (three of a kind, full house, straight, ...).

Video Poker II

Mathematics:

Define for all sets B of hands b containing 5 cards:

$$\mu(B) = \sum_{b \in B} \text{payoff}(b)$$

The main part of the computation deals with the **optimisation of the hold strategy** after the first draw. Compute to each of the

♦ approx. 2.6 mill. hands

the maximal expectation of payoff depending on the

♦ 32 possible decisions of holding 0 to 5 cards.

These 32 conditional expectations depend only on the hold cards k_1, ..., k_s and the removed cards k_{s+1}, ..., k_5:

$$\mu(\{B \mid k_1,\ldots,k_s \in B \wedge k_{s+1},\ldots,k_5 \notin B\}) / \binom{47}{5-s}$$

As a consequence of identities like

$$\mu(\{B \mid k_1,\ldots,k_4 \in B \wedge k_5 \notin B\}) = \mu(\{B \mid k_1,\ldots,k_4 \in B\}) - \mu(\{\{k_1,\ldots,k_5\}\})$$

$$\begin{aligned}\mu(\{B \mid k_1,\ldots,k_3 \in B \wedge k_4, k_5 \notin B\}) = {} & \mu(\{B \mid k_1,\ldots,k_3 \in B\}) \\ & - \mu(\{B \mid k_1,\ldots,k_3 \in B \wedge k_4 \notin B\}) \\ & - \mu(\{B \mid k_1,\ldots,k_3 \in B \wedge k_5 \notin B\}) \\ & + \mu(\{\{k_1,\ldots,k_5\}\})\end{aligned}$$

etc.

in a first step only sums of the form

$$\mu(\{B \mid k_1,\ldots,k_s \in B\})$$

must be computed (each of them corresponding to a conditional expectation "with replacement" of the removed cards).

Number Lotteries

„6 of 49“: 14 millions (exact: 13983816) possible combinations

wins: (approx.) 50% of the sum of all stakes is used as pay out – so lottery is a game against the other players

important: don’t choose combinations of numbers which are used often by other players (dates of birth, regular patterns, results of previous draws)

Frequencies of chosen combinations

(Karl Bosch, 6.8 mill. by players chosen combinations, Baden-Württemberg, 1993), expected frequency: 0.49:

1	2	3	4	5	6	~~7~~
8	9	10	11	12	~~13~~	14
15	16	17	18	~~19~~	20	21
22	23	24	~~25~~	26	27	28
29	30	~~31~~	32	33	34	35
36	~~37~~	38	39	40	41	42
43	44	45	46	47	48	49

4004

1	2	3	4	5	6	~~7~~
8	9	10	11	12	13	~~14~~
15	16	17	18	19	20	~~21~~
22	23	24	25	26	27	~~28~~
29	30	31	32	33	34	~~35~~
36	37	38	39	40	41	~~42~~
43	44	45	46	47	48	49

3817

1	2	3	4	~~5~~	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	~~27~~	28
29	30	31	32	33	~~34~~	~~35~~
36	~~37~~	38	39	40	41	42
43	44	45	46	47	48	~~49~~

3698

~~1~~	~~2~~	~~3~~	~~4~~	~~5~~	~~6~~	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

3249

1	2	3	~~4~~	5	6	7
8	9	10	~~11~~	12	13	14
15	16	17	~~18~~	19	20	21
22	23	24	~~25~~	26	27	28
29	30	31	~~32~~	33	34	35
36	37	38	~~39~~	40	41	42
43	44	45	46	47	48	49

2821

1	2	3	4	5	6	7
8	9	10	11	12	~~13~~	14
15	16	17	18	~~19~~	20	21
22	23	24	~~25~~	26	27	28
29	30	~~31~~	32	33	34	35
36	~~37~~	38	39	40	41	42
~~43~~	44	45	46	47	48	49

2335

Combinatorial games: theoretical

1912 **Zermelo**:
minimax theorem for chess etc.:
“Each position has a value -1, 0 or 1“
not valid: paper-stone-scissors,
3-person chess
generalisation possible for:
finite 2-person zero-sum games with perfect information (valid for expectations if there are chance elements included)

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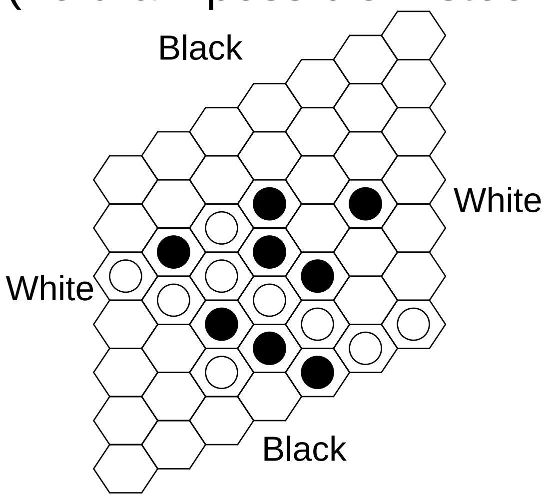
ca. 1948 **Nash**:
First player has a winning strategy for „Hex“
(no draw possible + steeling of strategy)



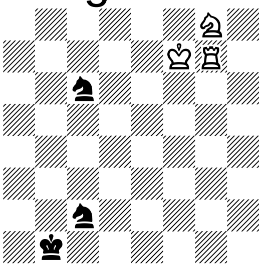
1979 **Reisch**:
Hex and Go-Moku (“Five in a row”) are PSPACE-complete (and therefore polynomial equivalent)

Combinatorial games: practical

1945-1950 **Turing**, **Shannon** (and **Zuse**):
Computer-chess: principle ideas and a first game (Turing),
tree search with minimax

circ. 1960 **Newell, Shaw, Simon** et.al.:
Computer-chess: “α-β“ doubles the search depth per computation time

1970-1996 **Ströhlein**, **Thompson**, **Stiller**:
Databases for several types of chess endgames:



White has a winning strategy, but he needs up to 243(×2) moves to capture a black knight

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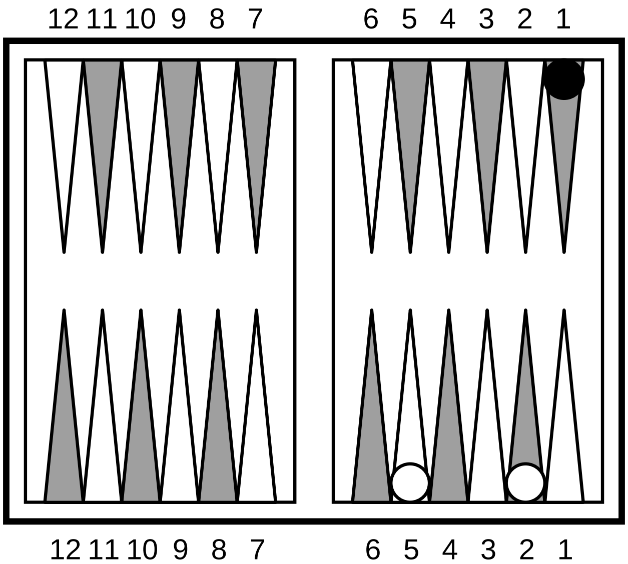
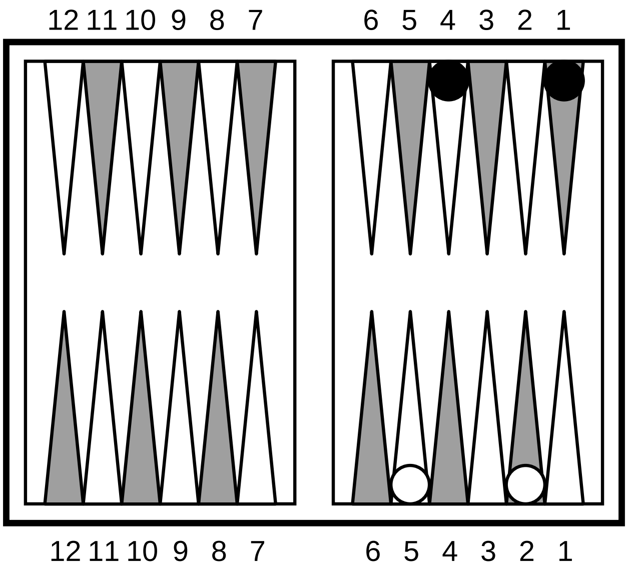
1993 **Allis**, **van den Herik**, **Huntjens**:
First player has a winning strategy at Go-Moku with a board size of $\geq$ 15×15.

1994 **Nievergelt**, **Gasser**:
No one has to loose at „Nine men’s morris”

Backgammon: Jacoby-Paradox



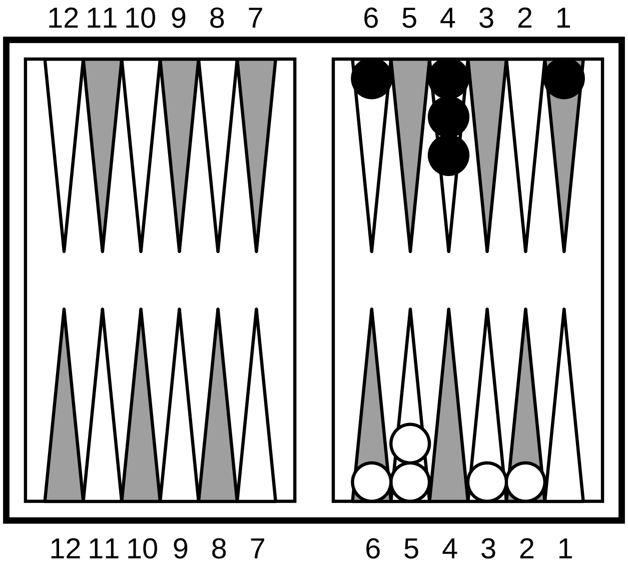
White to move: Double: „Yes“, **Redouble: „Yes”**
(Black should accept)



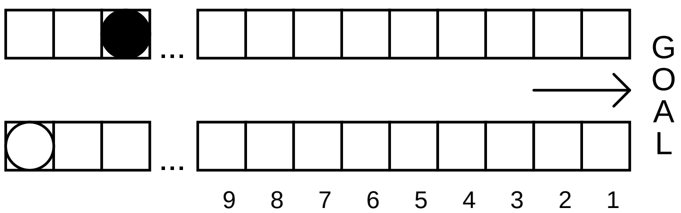
White to move: Double: „Yes“, **Redouble: „No“**
(Black should accept)

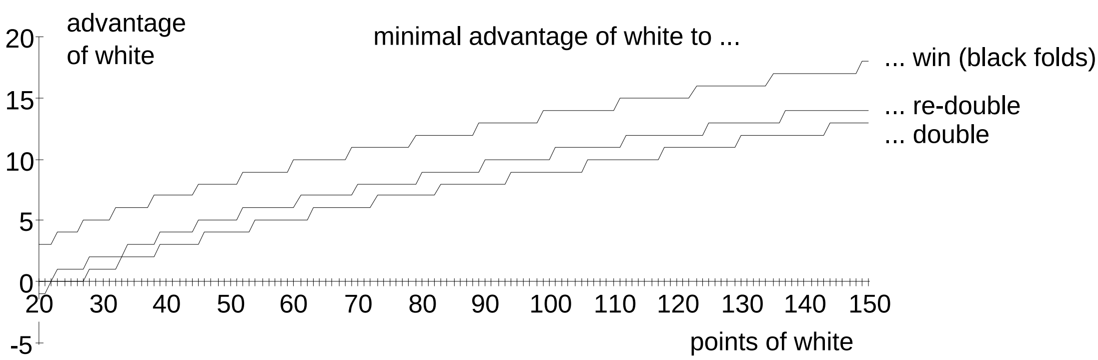
Conclusion: The probability that White wins is better in the second position. Nevertheless White should act more defensive in this second position (to wait for a better chance before his next move).

Backgammon: The running game ...



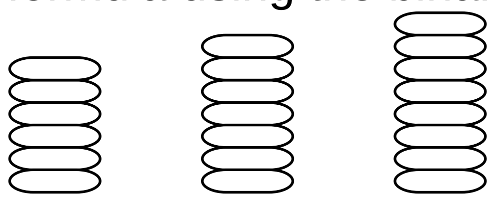
... and a model with two men





Combinatorial game theory

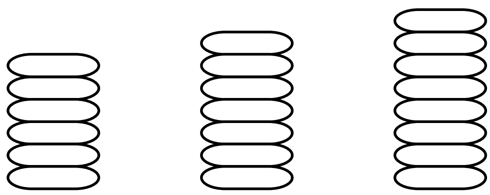
1901 **Bouton**
solved Nim, a impartial LPW-game
(LPW: last player wins), based on
a formula using the binary operation XOR



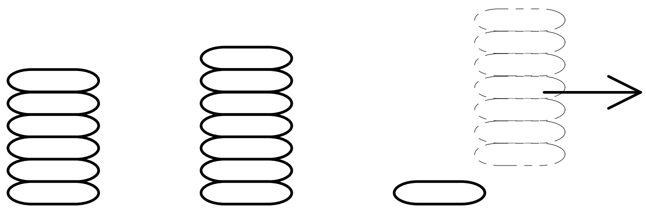
possible moves:
- select a pile
- remove men from this pile, how many you like (at minimum 1)

Move in such a way that the binary sum without carry (XOR) of the heap sizes is 0:

$6 +_2 7 +_2 8 = 9$



$6 +_2 9 = 15 > 6$ $7 +_2 9 = 14 > 7$ $8 +_2 9 = 1 < 8$



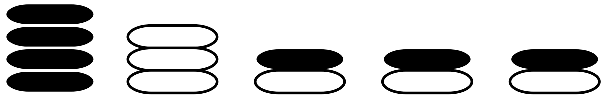
$6 +_2 7 +_2 1 = 0$

1931-1939 **Lasker**, **Sprague**, **Grundy**
analysed Nim-variants (impartial, LPW, mostly as disjunctive sums):
“Each position is equivalent to a Nim-position (i.e. replaceable in sums)“

1970- **Conway, Berlekamp, Guy**
partial games two-person-LPW-games;
constructions of real and transfinite numbers

LPW-game Black-White-Nim (simplified “Hackenbush” of **Conway, Berlekamp** and **Guy**)

Who can force a win, if White moves first, and who, if Black moves first?

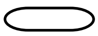


possible moves:
- select a stone of your color
- remove it and all stones above

The answer can be given using a homomorphism of ordered groups, defined as a “measure of advantage”. It can be interpreted as a generalisation of the

♦ Number of moves, White can play longer than Black:

{positions of Black-White-Nim} → **Q**,

 → 1,  → -1,  → 0,

 → 0,  → ½

Numbers	**Games**
addition	disjunctive sum: place the positions beside
inverse element	swap colours of all men
greater than 0	White has a winning strategy moving first or second
kernel	“0-positions” (second player has a winning strategy)
image in Q	fractions with powers of 2 as denominator

Therefore the first position is equivalent to

In consequence White, moving first or second, has a winning strategy.

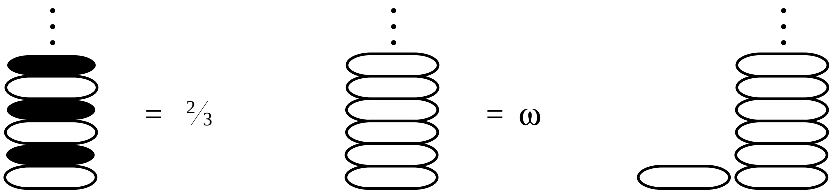
Black-White-Nim II

More examples (to be analysed recursively):

$\{0 \mid \}$	$\{0 \mid 1\}$	$\{0, \frac{1}{2} \mid 1\}$	$\{0, \frac{1}{2} \mid 1, \frac{3}{4}\}$	$\{0, \frac{1}{2} \mid 1, \frac{3}{4}, \frac{5}{8}\}$
$= 1$	$= \frac{1}{2}$	$= \frac{3}{4}$	$= \frac{5}{8}$	$= \frac{9}{16}$

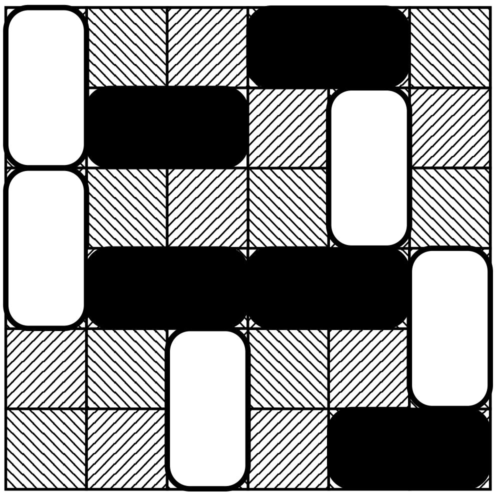
The notation {a, b, ... | u, v ...} describes the move options of White (a, b, ...) and Black (u, v, ...).

More of mathematical interest:
Using infinite towers we get more numbers (all possible sequences of moves are finite!):



$= \frac{2}{3}$ $\qquad = \omega$ $\qquad = \omega + 1$

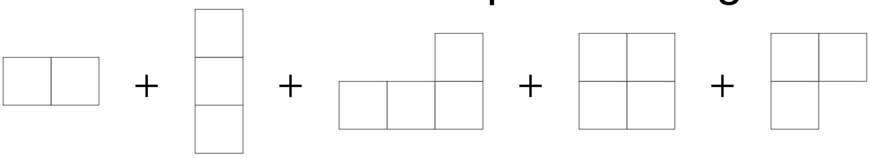
A LPW-game with dominoes:



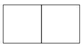
possible moves:
- White places a domino vertically
- Black places a domino horizontally

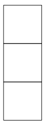
(so each move removes two squares)

Disjunctive sums are a natural part of the game:

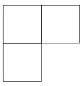


There are gaps which are not equivalent to a number:

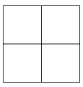
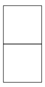
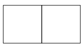
 $= \{ \mid 0 \} = -1$

 $= \{$  $\mid \} = \{ 0 \mid \} = 1$

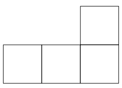
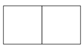
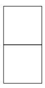
Every number $\varepsilon > 0$ fulfils ...

 $= \{$  $\mid$  $\} = \{ 0 \mid 0 \} =: \ast$

$\ast > -\varepsilon$
$\ast < \varepsilon$

 $= \{$  $\mid$  $\} = \{1 \mid -1\} =: \pm 1$

$\pm 1 > -1 - \varepsilon$
$\pm 1 < 1 + \varepsilon$

 $= \{$  $\mid$  $+$  $,$  $\} = \{ -1 \mid 0, 1\} = -\frac{1}{2}$

$$-1 + 1 - \tfrac{1}{2} + (\pm 1) + \ast = -\tfrac{1}{2} + \ast + (\pm 1)$$

The first moving player has a winning strategy:
- ♦ White moves to $-\frac{1}{2} + \ast + (+1) = \frac{1}{2} + \{0 \mid 0\}$ resp.
- ♦ Black moves to $-\frac{1}{2} + \ast - 1 = -1\frac{1}{2} + \{0 \mid 0\}$.

Strategic games, general games

Position: the actual state of a game (what happened in the past?), e.g. for card games: the hands of the players.

Information set: the subjective state of a game as seen by the player who has to move (corresponding to a set of positions), e.g. the player's own cards, not the opponent's one.

Move: based on his actual information a player acts, i.e. he selects one of the possible moves.

Strategy / pure strategy: a complete behaviour plan of a player which includes decisions for all his information sets (i.e. his subjective information states).

Mixed strategy: a player decides to choose his strategy in a random manner based on a probability distribution

Normal form of a game: function of payoffs to the players depending on the chosen (pure) strategies; in the case of a 2-person zero-sum game the normal form is simply a (mostly very big) matrix ("**matrix game**")

The main result (a generalisation of Zermelo):

1926/1928	**von Neumann** proved minimax theorem: "Each finite 2-person zero-sum game has a well defined value if it is played with mixed strategies". Poker model (publ.1944): bluffs can be seen as part of an objectively optimal behaviour

Strategic games – More results, history

1713	**Waldegrave**, **Montmort**, **N. Bernoulli**: mixed minimax behaviour for the 2-person card-game „Le Her“
1920-1924	**É. Borel**: 2-person-games: normal form, mixed strategies (e.g. draw with 5 at Baccarat?). Asked for optimal behaviour in symmetric games; proved a minimax theorem for symmetric 5x5-games, but doubted in a generalisation
1934	**R. A. Fisher**: mixed strategy as solution of „Le Her“ (independent from predecessors)
1944	**von Neumann**, **Morgenstern**: Foundation of game theory (games as a model of economic behaviour) Examples: Chess, Poker, Bridge
1950	**Nash** (Ph.D. thesis, Nobel price 1994): Existence of a Nash equilibrium: “If reached no single player is interested to change his mixed strategy”; his example: 3-pers. Poker
1953	**Kuhn**: In games with perfect recall it is sufficient to use behaviour strategies: mixtures are managed “local” by a random choice of moves. In the case of a Poker model already practically used in 1944 (von Neumann).

Mastermind

k^n-version: n colours, k pegs

rules: a **code maker** selects one of the k^n codes as hidden code; the **code breaker** guesses step by step one code and gets a feedback consisting

- the number of correctly guessed pegs
- the maximal number of correct pegs which can be reached with a permutation

positions: characterised by the set of all actual possible codes

Optimisation of strategy in Mastermind: Possibilities and results in the case of the 6^4-version:

- **worst case:**
  Code maker is allowed to “cheat“, i.e. he may change his code during the game (compatible to his previous feedback) – equivalent to 2-person-game with perfect information:
  5 guesses are enough (**Knuth** 1976).

- **average case:**
  Uniform distribution of codes is assumed.
  Backward induction gives optimal strategy for searching:
  4.340 (max. 6) guesses (**Koyama**, **Lai** 1993)

- **Minimax** (in the sense of a 2-personen-game)**:**
  Both players can use mixed strategies:
  $\leq$ 4.3674 (**Flood** 1986), $\geq$ 4.340 (see above)
  = 4.341 (**Wiener**, 1995, internet announcement)

A simple Poker model (2 players)

<u>The rules are symmetricly constructed:</u>

- ♦ 2 decks of 6 cards “1”, “2”, ... “6”, one for each player (the probabilities are uniformly distributed and independent)
- ♦ simultaneous bidding; allowed bets: 1,2,3, 5, 10 or 15.

<u>Normal form</u>: $6^6 = 46646$ pure strategies

Mimimax behaviour strategy:

bet \ hand	1	2	3	4	5	6
1	0,35857	0,56071	0,50643	0,46857		
2	0,33786	0,12179	0,41179			
3	0,14143	0,16500		0,51571	0,00429	
5	0,05629	0,12757			0,59286	
10	0,06700	0,02493	0,08179	0,01571	0,14029	
15	0,03886				0,26257	1,00000

"Shadow prices" (costs of a mistake):

bet \ hand	1	2	3	4	5	6
1					-0,18190	-3,33833
2				-0,02524	-0,28524	-3,44167
3			-0,09536			-3,15429
5			-0,07155	-0,23405		-2,66238
10						-2,92262
15		-0,05607	-0,23393	-0,39643		

How to **compute** optimal strategies **step by step**:
Choose a selection of pure strategies, compute a pair of relative minimax strategies and optimal response strategies. Append these response strategies to the selections if they aren’t included (otherwise the relative minimax strategies are optimal in total; 2×44 pure strategies are enough).

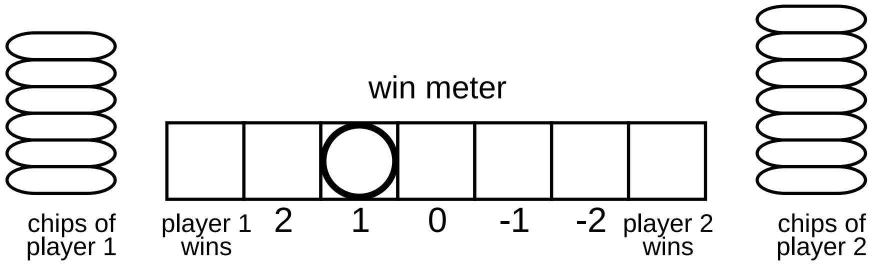
alternative: Linear programming with constraints
(**Romanovski** 1962, **von Stengel** 1996).

„QUAAK!“ – a game only for children?

Symmetric game for 2 players (publ.: Ravensburger):

- 2 players play several rounds,
- each player starts with 15 chips,
- in each round a player can bet 0 to 3 chips of his stock,
- the player with the higher bet wins the round
- A player who has won 3 more rounds than his opponent wins the game.

Recursive game with a normal form of size (until) 3×3 to every possible state, e.g.:



Starting position: (2×15 chips):

For the first move the optimal behaviour in the sense of a minimax strategy is:

0 Chips: 0.1212
1 Chip: 0.2272
2 Chips: 0.0000 (“shadow price“: 0.0852)
3 Chips: 0.6515

What can you get with an “optimal” strategy?

Take a **symmetric game** (e.g. a symmetrization of a non-symmetric game):

- ♦ Chess
- ♦ Backgammon
- ♦ Paper-stone-scissors, poker with two players
- ♦ a chess variant for three players
- ♦ poker with three players

No player can have a guaranty for a win of *more* than 0 (because the symmetry). But: Can you play in such a way that you have guarantied a win of 0 at minimum?

- ♦ **Chess**: Yes.
  A player who doesn’t make a bad move has a guaranty to win 0 at minimum.

- ♦ **Backgammon**: Yes (an average of 0 is guarantied if a player always moves optimal).

- ♦ **Paper-stone-scissors, Poker with two persons**: Yes (an average of 0 is guarantied if a player mixes his strategy in an optimal way).

- ♦ A **chess variant for three persons**: No.
  So it can’t be used as an intellectual competition.
  [There are only strategies which form an equilibrium].

- ♦ **Poker with three persons**: No.
  [There are only mixed strategies which form an equilibrium].