# Luck, Logic, and White Lies

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## The Mathematics of Games

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Wellesley, Massachusetts

Editorial, Sales, and Customer Service Office

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888 Worcester Street, Suite 230
Wellesley, MA 02482
www.akpeters.com



**Library of Congress Cataloging-in-Publication Data**

Bewersdorff, Jörg.
[Luck, logik und Bluff. English]
Luck, logic, and white lies : the mathematics of games / Jörg Bewersdorff ; translated by David Kramer.
p. cm.
Includes bibliographical references and index.
ISBN 1-56881-210-8
1. Game theory. I. Title

QA269.B39413 2004
519.3—dc22
2004053374

Printed in Canada

09 08 07 06 05 10 9 8 7 6 5 4 3 2 1

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# Preface

*A feeling of adventure is an element of games. We compete against the uncertainty of fate, and experience how we grab hold of it through our own efforts.* —Alex Randolph, game author

## The Uncertainty of Games

Why do we play games? What causes people to play games for hours on end? Why are we not bored playing the same game over and over again? And is it really the same game? When we play a game again and again, only the rules remain the same. The course of the game and its outcome change each time we play. The future remains in darkness, just as in real life, or in a novel, a movie, or a sporting event. That is what keeps things entertaining and generates excitement.

The excitement is heightened by the possibility of winning. Every player wants to win, whether to make a profit, experience a brief moment of joy, or have a feeling of accomplishment. Whatever the reason, every player can hope for victory. Even a loser can rekindle hope that the next round will bring success. In this, the hope of winning can often blind a player to what is in reality a small probability of success. The popularity of casino games and lotteries proves this point again and again.

Amusement and hope of winning have the same basis: the variety that exists in a game. It keeps the players guessing for a long time as to how the game will develop and what the final outcome will be. What causes this uncertainty? What are the mechanisms at work? In comparing games like roulette, chess, and poker, we see that there are three main types of mechanism:

1. chance;

2. the large number of combinations of different moves;

3. different states of information among the individual players.

Random influences occur in games involving dice and the mixing of cards. The course of a game, in accordance with its rules, is determined not only by decisions made by the players, but by the results of random processes. If the influence of chance dominates the decisions of the players, then one speaks of games of chance. In games of pure chance, the decision of a player to take part and the size of a player's bet are perhaps his[1] most important decisions. Games of chance that are played for money are generally governed by legal statute.

During the course of most games, there are certain situations in which the players have the opportunity to make decisions. The available choices are limited by the rules of the game. A segment of a game that encompasses just one such decision of a single player is called a *move*. After only a small number of moves, the number of possibilities can already represent an enormous number of combinations, a number so large that it is difficult to recognize the consequences of an individual move. Games whose uncertainty rests on the multiplicity of possible moves are called *combinatorial games*. Well-known representatives of this class are chess, go, nine men's morris, checkers, halma, and reversi. Games that include both combinatorial and random elements are backgammon and pachisi, where the combinatorial character of backgammon is stronger than that of pachisi.

A third source of uncertainty for the players of a game arises when the players do not all have the same information about the current state of the game, so that one player may not possess all the information that is available to the totality of players. Thus, for example, a poker player must make decisions without knowing his opponents' cards. One could also argue that in backgammon a player has to move without knowing the future rolls of the dice. Yet there is a great difference between poker and backgammon: no player knows what the future rolls of the dice will be, while a portion of the cards dealt to the players are known by each player. Games in which the players' uncertainty arises primarily from such imperfect information are called *strategic games*. These games seldom exist in a form that one might call purely strategic. Imperfect information is an important component of most card games, like poker, skat, and bridge. In the board games ghosts and Stratego, the imperfect information is based on the fact that one knows the location, but not the type, of the opponent's pieces.[2] In

[1]Translator's note: the German word for player, *Spieler*, is masculine, and so the author of this book could easily write the equivalent of "a player... his move" without too many qualms. Faced with this problem in English, I have decided to stick primarily with the unmarked masculine pronoun, with an occasional "his or her" lest the reader forget that both men and women, boys and girls, can play games.

[2]Ghosts and Stratego are board games for two players in which each player sees only the blank reverse side of his opponent's pieces. At the start, a player knows only his own pieces and the positions of the opposing pieces. In ghosts, which is played on a


**Figure P.1.** The three causes of uncertainty in games: a player wins through some combination of chance, logic, and bluff.

Diplomacy,[3] and rock–paper–scissors, [4] the players move simultaneously, so that each player is lacking the information about his opponent's current move. How this imperfect information plays out in a game can be shown by considering what happens to the game if the rules are changed so that the game becomes one of perfect information. In card games, the players would have to show their hands. Poker would become a farce, while skat would remain a combinatorially interesting game similar to the half-open two-person variant. In addition to the game rock–paper–scissors, which is a purely strategic game, poker is also recognized as a primarily strategic game. The degrees of influence of the three causes of uncertainty on various games are shown in Figure P.1.

There remains the question whether the uncertainty about the further course of the game can be based on other, as yet unknown, factors. If one investigates a number of games in search of such causes, one generally finds the following:

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chessboard with four good and four bad ghosts on the two sides, only the captured figures are revealed. In Stratego, the capturing power of a piece depends on its military rank. Therefore, a piece must be revealed to the opponent at the time of an exchange.

The simple rules of ghosts and a game with commentary can be found in *Spielbox* **3** 1984, pp. 37–39. Tactical advice on Stratego can be found in *Spielbox* **2** 1983, pp. 37 f.

[3]Diplomacy is a classic among board games. It was invented in 1945 by Alan Calhamer. Under the influence of agreements that the players may make among themselves, players attempt to control regions of the board, which represents Europe before the First World War. The special nature of Diplomacy is that the making and abrogating of agreements can be done secretly against a third party. An overview of Diplomacy appears in *Spielbox* **2** 1983, pp. 8–10, as well as a chapter by its inventor in David Pritchard (ed.), *Modern Board Games*, London 1975, pp. 26–44.

[4]Two players decide independently and simultaneously among the three alternatives "rock," "paper," and "scissors." If both players made the same choice, then the game is a draw. Otherwise, "rock" beats (breaks) "scissors," "paper" beats (wraps) "rock," and "scissors" beat (cut) "paper."

- the result of a game can depend on physical skill and performance ability. In addition to sports and computer games, which do not belong to the class of parlor games that we are considering here, Mikado is a game that requires manual dexterity.

- the rules of a game can be partially ambiguous. One arrives at such situations particularly in the learning phase of a complex game. In other cases, doubts arise in the natural course of the game. Thus in the crossword game Scrabble it can be unclear whether a word should be permitted. And even in skat, there are frequently questions raised, if only about minor details.

- an imperfect memory does not make only the game "memory" more difficult. However, this type of uncertainty is not an objective property of the game itself.

In comparison to chance, combinatorial richness, and differing informational states, these last phenomena can safely be ignored. None of them can be considered a typical and objective cause of uncertainty in a parlor game.

## Games and Mathematics

If a player wishes to improve his prospects of winning, he must first attempt to overcome his degree of uncertainty as much as possible and then weigh the consequences of his possible courses of action. How that is to be managed depends, of course, on the actual causes of the uncertainty: if a player wishes to decide, for example, whether he should take part in a game of chance, then he must first weigh the odds to see whether they are attractive in comparison to the amount to be wagered. A chess player, on the other hand, should check all possible countermoves to the move he has in mind and come up with at least one good reply to each of them. A poker player must attempt to determine whether the high bid of his opponent is based on a good hand or whether it is simply a bluff. All three problems can be solved during a real game only on a case-by-case basis, but they can also be investigated theoretically at a general level. In this book, we shall introduce the mathematical methods that have been developed for this and provide a number of simple examples:

- games of chance can be analyzed with the help of probability theory. This mathematical discipline, which today is used in a variety of settings in the natural sciences, economics, and the social sciences, grew out of a $17^{\text{th}}$-century desire to calculate the odds in a game of chance.

- there is no unified theory for the combinatorial elements in games. Nonetheless, a variety of mathematical methods can be used for answering general questions as well as solving particular problems.

- out of the strategic components of games there arose a separate mathematical discipline, called *game theory*, in which games serve as a model for the investigation of decison-making in interactive economic processes.

For all three game types and their mathematical methods, the computer has made possible applications that formerly would have been unthinkable. But even outside of the development of ever faster computers, the mathematical theory itself has made great strides in the last hundred years. That may surprise those unversed in modern mathematics, for mathematics, despite a reputation to the contrary, is by no means a field of human endeavor whose glory days are behind it.

*Probability theory* asks questions such as, which player in a game of chance has the best odds of winning? The central notion is that of *probability*, which can be interpreted as a measure of the certainty with which a random event occurs. For games of chance, of course, the event of interest is that a particular player wins. However, frequently the question is not who wins, but the amount of the winner's winnings, or score. We must then calculate the average score and the risk of loss associated with it. It is not always necessary to analyze a game completely, for example, if we wish only to weigh certain choices of move against each other and we can do so by a direct comparison. In racing games governed by dice, one can ask questions like, how long does it take on average for a playing piece to cover a certain distance? Such questions can become complicated in games like snakes and ladders, in which a piece can have the misfortune to slip backward. Even such a question as which squares in the game Monopoly are better than others requires related calculational techniques. It is also difficult to analyze games of chance that contain strong combinatorial elements. Such difficulties were first overcome in the analysis of blackjack.

*Combinatorial games*, such as the tradition-rich chess and go, are considered games with a high intellectual content. It was quite early in the history of computational machines that the desire was expressed to develop machines that could serve as worthy opponents in such games. But how could that be accomplished? Indeed, we need computational procedures that make it possible to find good moves. Can the value of a move be somehow uniquely determined, or does it always depend on the opponent's reply? In any case, the current state of technology for search procedures and computational techniques is impressive. An average chess player no longer has a ghost of a chance against a good chess program. And it is not only chess that has been the object of mathematical interest. Winning strategies have been found for many games, some of them surprisingly simple. For other games it has been determined only which player theoretically should win, without a winning strategy actually being found. Some of these games possess properties that make it doubtful whether such a strategy will ever be found.

It is a task of *game theory* to determine how strategic games differ fundamentally from combinatorial games and games of chance. First, one needs a mathematical definition of a game. A game is characterized by its rules, which include the following specifications:

- the number of players.
- for each game state, the following information:
  - whose move it is;
  - the possible moves available to that player;
  - the information available to that player in deciding on his move.
- for games that are over, who has won.
- for random moves, the probabilities of the possible results.

Game theory arose as an independent discipline in 1944, when out of the void there appeared a monumental monograph on the theory of games. Although it mentions many popular games such as chess, bridge, and poker, such games serve game theory only as models of economic processes. It should not be surprising that parlor games can serve as models for real-life interactions. Many games have borrowed elements of real-life struggles for money, power, or even life itself. And so the study of interactions among individuals, be it in cooperation or competition, can be investigated by looking at the games that model those interactions. And it should come as no surprise that the conflicts that arise in the games that serve as models are idealized. That is just as inevitable as it is with other models, such as in physics, for example, where an object's mass is frequently considered to be concentrated at a single point.

## About This Book

We have divided the book into three parts to reflect our division of games into three types, and so we investigate mathematically in turn the chance, combinatorial, and strategic elements of games. Each of the three parts encompasses several chapters, each of which considers a specific problem—generally a game or game fragment.

In order to reach as broad an audience as possible, we have not sought the generality, formalism, and completeness that are usual in textbooks. We are more concerned with ideas, concepts, and techniques, which we discuss to the extent that they can be transferred to the study of other games.

Due to the problem-oriented selection of topics, the mathematical level differs widely among the different chapters. Although there are frequent references to earlier chapters, one can generally read each chapter independently of the others. Each chapter begins with a question, mostly of a rhetorical nature, that attempts to reveal the nature and difficulty of the problem to be dealt with. This structure will allow the more mathematically sophisticated readers, for whom the mathematical treatment will frequently be too superficial and incomplete, to select those parts of greater mathematical interest. There are many references to the specialist literature for those who wish to pursue an issue in greater depth. We have also given some quotations and indications of the mathematical background of a topic as well as related problems that go beyond the scope of the book.

We have placed considerable emphasis on the historical development of the subject, in part because recent developments in mathematics are less well known than their counterparts in the natural sciences, and also because it is interesting to see how human error and the triumph of discovery fit into a picture that might otherwise seem an uninterrupted sequence of great leaps forward. The significance of the progress of mathematics, especially in recent decades, in the not necessarily representative area of game theory, can be seen by a comparison with thematically similar, though often differing in detail of focus, compilations that appeared before the discovery of many of the results presented in this book:

- René de Possel, *Sur la théorie mathématique des jeux de hasard et de réflexion*, Paris 1936. Reprinted in Hevre Moulin, *Fondation de la théorie des jeux*, Paris 1979.

- R. Vogelsang, *Die mathematische Theorie der Spiele*, Bonn 1963.

- N. N. Vorob'ev, *The Development of Game Theory* (in Russian), 1973. The principal topic is game theory as a mathematical discipline, but this book also contains a section on the historical development of the theories of games of chance, and combinatorial and strategic games.[5]

- Richard A. Epstein, *The Theory of Gambling and Statistical Logic*, New York 1967 (expanded revised edition, 1977).
- Edward Packel, *The Mathematics of Games and Gambling*, Washington 1981.
- John D. Beasley, *The Mathematics of Games*, Oxford 1989.
- *La mathématique des jeux, Bibliothèque pour la Science*, Paris 1997. Contributions on the subject of games from the French edition of *Scientific American*, some of which have been published in the editions of other countries.

## Acknowledgments

I would like to express my thanks to all those who helped in the development of this book: Elwyn Berlekamp, Richard Bishop, Olof Hanner, Julian Henny, Daphne Koller, Martin Müller, Bernhard von Stengel, and Baris Tan kindly explained to me the results of their research. I would like to thank Bernhard von Stengel additionally for some remarks and suggestions for improvements and also for his encouragement to have this book published. I would also like to thank the staffs of various libraries who assisted me in working with the large number of publications that were consulted in the preparation of this book. As representative I will mention the libraries of the Mathematical Institute of Bonn, the Institute for Discrete Mathematics in Bonn, as well as the university libraries in Bonn and Bielefeld. Frauke Schindler, of the editorial staff of Vieweg-Verlag, and Karin Buckler have greatly helped in minimizing the number of errors committed by me. I wish to thank the program director of Vieweg-Verlag, Ulrike Schmickler-Hirzebruch, for admitting into their publishing progam a book that falls outside of the usual scope of their publications. Last but not least, I thank my wife, Claudia, whose understanding and patience have at times in the last several years been sorely taxed.

[5]The author wishes furthermore to express his gratitude for observations of Vorob'ev for important insights that have found their way into this preface.

## Preface to the Second Edition

The happy state of affairs that the first edition has sold out after two years has given me the opportunity to correct a number of errors. Moreover, I have been able to augment the references to the literature and to newer research. I wish to thank Hans Riedwyl, Jürg Nievergelt, and Avierzi S. Fraenkel for their comments.

Finally, I wish to direct readers to my web site, http://www.bewersdorff-online.de/, for corrections and comments.

## Preface to the Third Edition

Again, I have alert readers to thank for calling a number of errors to my attention: Pierre Basieux, Ingo Briese, Dagmar Hortmeyer, Jörg Klute, Norbert Marrek, Ralph Rothemund, Robert Schnitter, and Alexander Steinhansens. In this regard I would like to offer special thanks to David Kramer, who found errors both logical and typographical in the process of translating this book into English.

The necessity to make changes to the content of the book arose from some recently published work, especially that of Dean Allemang on the misère version of nim games, and of Elwyn Berlekamp on the idea of environmental go. I have also gladly complied with the request of readers to include newer approaches to game tree search procedures.

—Jörg Bewersdorff[6]

[6]I would be glad to receive comments on errors and problems with the text at joerg.bewersdorff@t-online.de. Questions are also welcome, and I will answer them gladly within the constraints of time and other commitments.

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